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Diabolical points in the resonance spectra of vibrating smectic films

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Recent theoretical studies reveal the existence of so-called diabolical points in the energetic spectra of rectangular quantum billiards with a pointlike scatterer. The wave equation that rules the drum-head oscillations of free-standing smectic films is similar to the two-dimensional Schrödinger equation, which makes vibrating smectic films the analogues of appropriate quantum billiards. In this Rapid Communication we study experimentally the diabolical points in a family of rectangular quantum billiards with an infinite scatterer via the resonance spectra of appropriate smectic films. [S1063-651X(98)52010-7]

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The name *diabolical point* refers to a type of degeneracy in the energy spectra of families of quantum systems without symmetry and for which no magnetic field is present [1]. They appear systematically in families that are characterized by at least two parameters. The energy surfaces represented in the space of these two parameters are connected only at isolated points, and the energy surfaces in the neighborhood of the connection point exhibit a conical geometry as a diabolo [2], which is responsible for the name of these intersections (Fig. 1).

In particular, the degeneracies of this type appear in the spectra of so-called *quantum billiards* [3]. The wave functions $\psi(x,y)$ of the quantum particle of the billiard as well as the energy levels E are given by the two-dimensional Schrödinger equation

$$-\Delta\psi(x,y)=E\psi(x,y),\quad \psi|_{\mathrm{boundary}}=0. \tag{1}$$

Equation (1) is analogous to the wave equation that rules vibration of a free-standing smectic film [4] characterized by a constant two-dimensional density $\rho_{2\mathrm{D}}$ and a tension τ [5] in the limit of small amplitudes A [6],

$$-\Delta(Az(x,y))=\frac{\rho_{2\mathrm{D}}(2\pi f)^2}{\tau}Az(x,y),\quad z|_{\mathrm{frame}}=0. \tag{2}$$

Here $z(x,y)$ and f are a stationary wave pattern and corresponding resonance frequency of the film, respectively.

This makes the smectic films analogous to quantum billiards in the same manner as acoustic [7] or microwave [8–10] cavities. The wave patterns $z(x,y)$ are equivalent to the wave functions of the corresponding quantum billiard $\psi(x,y)$, and the resonance frequencies f are related to the square roots of the energy levels E. Crossing of the energy levels of a quantum billiard corresponds to crossing of the resonance frequencies of the appropriate smectic film. Therefore, investigation of the resonance spectra of appropriate smectic films gives the experimental ground to the theoretical studies of quantum billiards.

To our knowledge, the existence of diabolical points in quantum billiards was first analyzed by Berry and Wilkinson for the family of triangular quantum billiards [1]. They pointed out theoretically and numerically the presence of diabolical points in this family of billiards. These predictions were verified experimentally using the microwave cavities [11].

Recently a number of theoretical and numerical studies was devoted to quantum billiards of rectangular shape with a pointlike scatterer of strength s situated in a point (x_0,y_0) [12–16]. Such a scatterer is described by a supplemental term to the Schrödinger equation (1)

$$[-\Delta+s\,\delta(x-x_0)\,\delta(y-y_0)]\psi(x,y)=E\psi(x,y),$$
$$\psi|_{\mathrm{boundary}}=0. \tag{3}$$

Once the aspect ratio $\alpha=L_y/L_x$ of the billiard and the scatterer strength s are fixed, the behavior of the billiard depends only on the coordinates of the scatterer (x_0,y_0) which play the role of control parameters. Therefore, the diabolical points should be the particular positions of the scatterer that induce degeneracies of the particular energy levels.

In [16] Cheon and Shigehara point out theoretically and numerically the existence of the diabolical points in the families of rectangular quantum billiards with a pointlike scatterer. They suggest to search these points along the nodal lines of unperturbed eigenstates of Eq. (1). A pointlike scatterer placed on a nodal line does not affect the corresponding

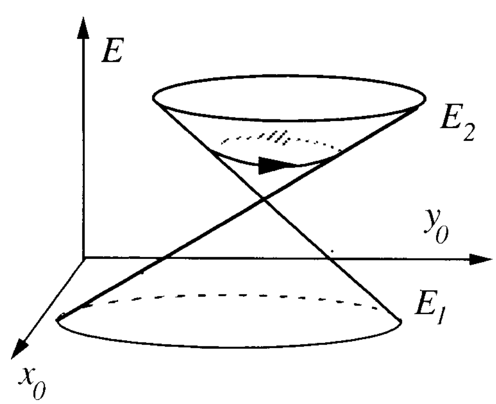

FIG. 1. Geometry of the energy surfaces in the space of two parameters (x_0,y_0) in the vicinity of a diabolical point.

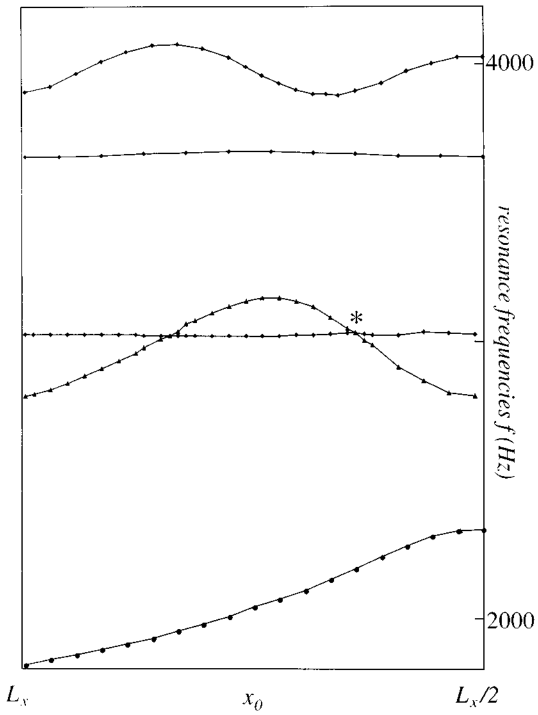

FIG. 2. Spectrum of a smectic film of dimensions 1.15:1 cm as a function of the x_0 coordinate of the fiber, whereas the y_0 coordinate of the fiber is $L_y/2$. Note the presence of two crossings between the second and the third levels $x_0 = 0.72$ cm (asterisk) and $x_0 = 0.94$.

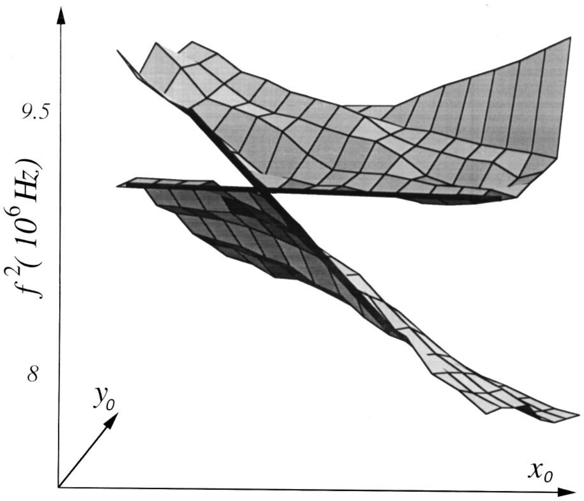

FIG. 3. Shape of the resonance surfaces in the vicinity of the diabolical point $x_0 = 0.72$ cm. The fact that the crossing point is situated on the mirror axis $y_0 = L_y/2$ permits us to examine only the area on one side of this axis determined by $y_0 \geqslant L_y/2$. The dimensions of the explored area are 0.7×1.5 mm.

energy level. However, the geometry of an adjacent state wave function is different and therefore its energy level is usually affected by the scatterer. If the perturbation is strong enough, the lower energy level reaches the unperturbed one [17], which leads to a diabolical point.

The numerical results in [16], performed for a family of rectangular billiards of an aspect ratio $\alpha \approx \pi/e$ with a strong negative scatterer [18], display a set of diabolical points situated on the line $x_0 = L_x/2$. In particular, the lowest energy levels are connected at two symmetric diabolical points [16].

Our aim here is to test experimentally the existence of diabolical points in this type of quantum billiards via the resonance spectra of vibrating smectic films. A scatterer is realized by a fine fiber suspended perpendicular to the film surface. If the diameter of the fiber d_{fib} is small enough with respect to the wavelength under consideration, it can be considered as a pointlike perturbation situated in a point (x_0, y_0). The viscous drag imposes a node in this point in all the wave patterns. This makes vibrating smectic film pierced by a fiber analogous to the rectangular quantum billiard of same dimensions with an infinite pointlike scatterer situated at the same point (x_0, y_0).

The quantitative data shown here were obtained on a Sm-C mixture at ambient temperature (SCE4 product from British Drug House). We use a frame of dimensions 1.15×1 cm whose aspect ratio is close to π/e taken for the numerical calculations in [16]. The film oscillation is driven by the electric field [6,5]. A small fiber of diameter $d_{\mathrm{fib}} = 0.3$ mm pierces the film surface in a point of coordinates (x_0, y_0) with respect to a corner of the frame. The precision of the reference point is of the order of 0.3 mm, whereas the coordinates (x_0, y_0) are controlled with the accuracy of 0.01 mm. The system is situated *in vacuo* to avoid the influence of the air surrounding the film.

A laser beam reflected from the vibrating surface is then sent to a photodiode connected to a lock-in amplifier which also delivers the excitation AC voltage. This method allows us to detect the resonance spectra of the film but not the shape of vibrating surface.

Equation (2) possesses an analytical solution that gives the resonance spectra and corresponding wave patterns of an unperturbed vibrating rectangular film:

$$f_{nm} = \frac{1}{2}\sqrt{\frac{\tau}{\rho_{2\mathrm{D}}}\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right)},$$
$$z_{nm}(x,y) = \sin\left(\pi n \frac{x}{L_x}\right)\sin\left(\pi m \frac{y}{L_y}\right). \tag{4}$$

First we tested the frequency shift due to the fiber in the lowest eigenmode with the intention of finding experimentally the diabolical point described in [16]. For this purpose, frequencies of the first and second eigenmodes are measured for a set of the fiber positions situated on the line $x_0 = L_x/2$. We find that the second resonance frequency remains constant, independent of the value of y_0, which confirms experimentally the approximation of the fiber as a pointlike object for the area of frequencies under consideration. As expected, the value of the first resonance frequency increases as the fiber approaches the center of the film. However, the value of the resonance frequency shifted by the fiber never reaches the second resonance frequency.

This fact shows the crucial importance of the value of the scatterer strength. The diabolical point generated by a negative scatterer disappears in the case of an infinite one. This is due to the fact that the influence of a negative scatterer is more important than the perturbation introduced by an infinite perturbation [15].

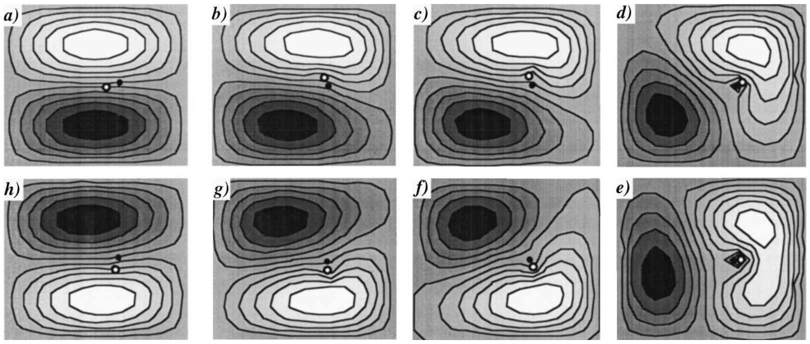

FIG. 4. (a)–(h). Rearrangement of the stationary wave patterns of the third mode on a closed loop including a crossing, which is labeled by the black point. The circles show the location of the fiber.

Let us note that the relative difference between the second and third unperturbed eigenfrequencies of Eq. (4) for the chosen aspect ratio L_y/L_x is much smaller than the distance between the first and the second ones. This fact makes possible the crossing of these resonance frequencies even in the case of an infinite perturbation simulated by a fiber. In this way, the search method from [16] requires us to explore the nodal line of the third mode situated on the line $y_0=L_y/2$. The eventual location of these diabolical points is not explored in [16].

Figure 2 shows the resonance spectra of the smectic film as a function of the x_0-fiber coordinate when the y_0 position is fixed to $L_y/2$. This set of fiber positions is situated also on the nodal line of the fourth unperturbed vibrating film mode. Accordingly, the experimental values of third and fourth frequencies remain unchanging all over the interval of x_0 under consideration.

We observe effectively two intersections between the second and third resonance frequencies situated at the points $x_0=0.72$ cm and $x_0=0.94$ cm (Fig. 2). In order to determine the nature of these crossings, we choose arbitrarily one of these intersections ($x_0=0.72$, $y_0=L_y/2$) for a more detailed analysis.

We have studied experimentally the behavior of the resonance frequencies in the vicinity of the crossing $x_0=0.72$ cm. The resonance surfaces are plotted in Fig. 3 [19]. This plot confirms the pointlike nature of the degeneracy. The conical geometry of two surfaces is not well pronounced, which is due to the imperfection of measurements when the two resonance frequencies are close to one another.

The eigenfrequencies f in Eq. (2) are given by

$$\sum_{n,m} z_{nm}^2(x_0,y_0)\frac{1}{4\pi^2(\rho_{2\mathrm{D}}L_x^2/\tau)(f^2-f_m^2)}=0. \tag{5}$$

Here, z_{nm} and f_{nm} are given by Eq. (4).

Equation (5) is not well defined with a full unperturbed basis $\{z_{nm}\}$. However, the finiteness of the real fiber induces a suitable cut-off of higher modes, which makes the problem well defined. The value of the cut-off n is then determined by the size of the fiber with respect to the film dimensions so that the wave vector of the cut-off $k_n d_{\mathrm{fib}}\approx a\pi,\ a\approx 1$.

We performed the numerical estimations from Eq. (5) with the use of 300 lowest eigenmodes. For the 300th mode $k_{300}\approx\sqrt{4\pi\times 300/L_xL_y}$ is of the order of 5.7 mm^{-1}, so that $k_{300}d_{\mathrm{fib}}\approx\pi/2$. This means that the real fiber behaves as a pointlike object for about 300 lowest modes. The simulations situate the crossing point at $x_0=0.74$ cm, which is in good agreement with the experimental results $x_0=0.72$ cm.

Let us note that even if the resonance spectra depends on the film tension and the two-dimensional density of a given film, the value $4\pi^2(\rho_{2\mathrm{D}}L_x^2/\tau)f^2$ (6) is a function only of the aspect ratio L_x/L_y.

Once the resonance frequency is found, the shape of vibrating film is given by

$$z(x,y)=A\sum_{n,m}\frac{z_{mn}(x_0,y_0)}{f^2-f_{nm}^2}\,z_{nm}(x,y), \tag{6}$$

where A is the amplitude of vibration.

An important characteristic of diabolical points is the so-called geometric, or Berry, phase effect. It means that in the space of control parameters on tracing a closed path, including a diabolical point, the wave function acquires an additional geometric phase π [20] (Fig. 1). This property is due to the singularity of the energy surfaces at the diabolical point and it enables one to distinguish diabolical points from other types of crossing.

As the wave function of a quantum billiard is analogous to the stationary wave pattern of the corresponding smectic film, the Berry phase test means that the wave pattern gains a supplemental phase π when the fiber is transported around a diabolical point. Since the experimental setup does not allow purely experimental realization of the test, it is possible to verify it with the use of Eq. (6). For this purpose, we have performed the numerical calculations of the film shape $z(x,y)$ for a set of fiber position situated on a closed path around the crossing using the experimental values of the resonance frequencies $f(x_0,y_0)$.

Figure 4 shows the results of a numerical test for the highest sheet of the resonance surfaces of Fig. 3. The 900 lowest modes are taken into account. Even if the influence of the finite size of the fiber [21] together with the measurements error induces a small shift of the nodal line from the fiber, the calculations give a decent representation of stationary patterns. As expected, the film shape changes the sign on a closed loop around the crossing point, which confirms the ''diabolical'' character of this intersection.

In conclusion, we report here the experiments on the resonance spectra of vibrating smectic films. We showed that vibrating smectic films are mathematically analogous to quantum billiards of appropriate shape, and a very small fiber on the film surface is equivalent to an infinite pointlike scatterer in the corresponding quantum billiard. A comparison between the recent theoretical studies and our experiments shows the crucial influence of the scatterer strength on the existence of diabolical points. We point out experimentally the location of two pointlike intersections between the second and the third energy levels. A detailed study of one of the intersections confirms that these intersections are diabolical points.

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